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AUTHOR(S):

Owa, Shigeyoshi; Yagi, Daisuke; Pascu, Nicolae N.

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Generalization Properties of Integral Means by H.Silverman

Shigeyoshi Owa, Daisuke Yagi and Nicolae N. Pascu

Abstract

For analytic and univalent functions $f(z)$ with negative coefficients in the open unit disk \mathbb{U} , H.Silverman (Houston J. math. **23**(1997)) has given some interesting results for integral means of $f(z)$. In the present paper, we discuss generalization properties of integral means of $f(z)$ given by H.Silverman. We also show some examples of our theorems.

1 Introduction

Let \mathcal{A} denote the class of functions $f(z)$ of the form

$$(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

that are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let \mathcal{S} be the subclass of \mathcal{A} consisting of all univalent functions $f(z)$ in \mathbb{U} . Also let \mathcal{S}^* and \mathcal{K} denote the subclasses of \mathcal{S} consisting of functions $f(z)$ which are starlike and convex in \mathbb{U} , respectively.

The class \mathcal{T} is defined as the subclass of \mathcal{S} consisting of all functions $f(z)$ which are given by

$$(1.2) \quad f(z) = z - \sum_{n=2}^{\infty} a_n z^n \quad (a_n \geq 0).$$

Further, we denote by $\mathcal{T}^* = \mathcal{S}^* \cap \mathcal{T}$ and $\mathcal{C} = \mathcal{K} \cap \mathcal{T}$. It is well-known by Silverman [3] that

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Remark 1.1. A function $f(z) \in \mathcal{T}^*$ if and only if

$$(1.3) \quad \sum_{n=2}^{\infty} na_n \leq 1.$$

A function $f(z) \in \mathcal{C}$ if and only if

$$(1.4) \quad \sum_{n=2}^{\infty} n^2 a_n \leq 1.$$

For $f(z) \in \mathcal{A}$ and $g(z) \in \mathcal{A}$, $f(z)$ is said to be subordinate to $g(z)$ in \mathbb{U} if there exists an analytic function $\omega(z)$ in \mathcal{U} such that $\omega(0) = 0$, $|\omega(z)| < 1$ ($z \in \mathbb{U}$), and $f(z) = g(\omega(z))$. We denote this subordination by

$$(1.5) \quad f(z) \prec g(z). \quad (\text{cf. Duren}[1])$$

For subordinations, Littlewood [2] has give the following integral mean.

Theorem A. If $f(z)$ and $g(z)$ are analytic in \mathbb{U} with $f(z) \prec g(z)$, then, for $\lambda > 0$ and $|z| = r$ ($0 < r < 1$),

$$(1.6) \quad \int_0^{2\pi} |f(re^{i\theta})|^\lambda d\theta \leq \int_0^{2\pi} |g(re^{i\theta})|^\lambda d\theta.$$

Furthermore, Silverman [3] has shown that

Remark 1.2. $f_1(z) = z$ and $f_n(z) = z - \frac{z^n}{n}$ ($n \geq 2$) are extreme points of the class \mathcal{T}^* (or \mathcal{T}). $f_1(z) = z$ and $f_n(z) = z - \frac{z^n}{n^2}$ ($n \geq 2$) are extreme points of the class \mathcal{C} .

Applying Theorem A with extreme points of \mathcal{T} , Silverman [4] has proved the follwing results.

Theorem B. Suppose that $f(z) \in \mathcal{T}$, $\lambda > 0$ and $f_2(z) = z - \frac{z^2}{2}$. Then, for $z = re^{i\theta}$ ($0 < r < 1$),

$$(1.7) \quad \int_0^{2\pi} |f(z)|^\lambda d\theta \leq \int_0^{2\pi} |f_2(z)|^\lambda d\theta.$$

Theorem C. If $f(z) \in \mathcal{T}$, $\lambda > 0$, and $f_2(z) = z - \frac{z^2}{2}$, then, for $z = re^{i\theta}$ ($0 < r < 1$),

$$(1.8) \quad \int_0^{2\pi} |f'(z)|^\lambda d\theta \leq \int_0^{2\pi} |f'_2(z)|^\lambda d\theta.$$

In the present paper, we consider the generalization properties for Theorem B and Theorem C with $f(z) \in \mathcal{T}^*$ and $f(z) \in \mathcal{C}$.

2 Generalization properties

Our first result for the generalization properties is contained in

Theorem 2.1. Let $f(z) \in \mathcal{T}^*$, $\lambda > 0$, and $f_k(z) = z - \frac{z^k}{k}$ ($k \geq 2$). If $f(z)$ satisfies

$$(2.1) \quad \sum_{j=0}^{k-3} \frac{j+1}{k} (a_{2k+j-1} + a_{k+j+1} - a_{k-j-1}) \geq 0$$

for $k \geq 3$, then, for $z = re^{i\theta}$ ($0 < r < 1$),

$$(2.2) \quad \int_0^{2\pi} |f(z)|^\lambda d\theta \leq \int_0^{2\pi} |f_k(z)|^\lambda d\theta.$$

Proof. For $f(z) \in \mathcal{T}^*$, we have to show that

$$\int_0^{2\pi} \left| 1 - \sum_{n=2}^{\infty} a_n z^{n-1} \right|^\lambda d\theta \leq \int_0^{2\pi} \left| 1 - \frac{z^{k-1}}{k} \right|^\lambda d\theta.$$

By Theorem A, it suffices to prove that

$$1 - \sum_{n=2}^{\infty} a_n z^{n-1} \prec 1 - \frac{z^{k-1}}{k}.$$

Let us define the function $\omega(z)$ by

$$(2.3) \quad 1 - \sum_{n=2}^{\infty} a_n z^{n-1} = 1 - \frac{1}{k} \omega(z)^{k-1}.$$

It follows from (2.3) that

$$|\omega(z)|^{k-1} = \left| k \sum_{n=2}^{\infty} a_n z^{n-1} \right| \leq |z| \left(\sum_{n=2}^{\infty} k a_n \right).$$

Thus, we only show that

$$\sum_{n=2}^{\infty} k a_n \leq \sum_{n=2}^{\infty} n a_n,$$

or

$$\sum_{n=2}^{\infty} a_n \leq \frac{1}{k} \left(\sum_{n=2}^{\infty} n a_n \right).$$

Indeed, we see that

$$\begin{aligned}
\frac{1}{k} \sum_{n=2}^{\infty} n a_n &= \left(1 - \frac{k-2}{k}\right) a_2 + \left(1 - \frac{k-3}{k}\right) a_3 + \cdots \\
&\quad + \left(1 - \frac{2}{k}\right) a_{k-2} + \left(1 - \frac{1}{k}\right) a_{k-1} + a_k + \left(1 + \frac{1}{k}\right) a_{k+1} \\
&\quad + \left(1 + \frac{2}{k}\right) a_{k+2} + \cdots + \left(1 + \frac{k+1}{k}\right) a_{2k+1} + \left(1 + \frac{k+2}{k}\right) a_{2k+2} \\
&\quad + \cdots \\
&= \frac{k-2}{k} (a_{2k-2} - a_2) + \frac{k-3}{k} (a_{2k-3} - a_3) + \cdots \\
&\quad + \frac{2}{k} (a_{k+2} - a_{k-2}) + \frac{1}{k} (a_{k+1} - a_{k-1}) \\
&\quad + \left(1 + \frac{k-1}{k}\right) a_{2k-1} + \left(1 + \frac{k}{k}\right) a_{2k} + \left(1 + \frac{k+1}{k}\right) a_{2k+1} \\
&\quad + \cdots + \sum_{n=2}^{2k-2} a_n.
\end{aligned}$$

Nothing that

$$1 + \frac{k+j}{k} \geq 1 + \frac{2+j}{k} \quad (j = -1, 0, 1, \dots),$$

we obtain

$$\begin{aligned}
(2.4) \quad \frac{1}{k} \left(\sum_{n=2}^{\infty} n a_n \right) &\geq \frac{k-2}{k} (a_{2k-2} - a_2) + \frac{k-3}{k} (a_{2k-3} - a_3) \\
&\quad + \cdots + \frac{2}{k} (a_{k+2} - a_{k-2}) + \frac{1}{k} (a_{k+1} - a_{k-1}) \\
&\quad + \left(1 + \frac{1}{k}\right) a_{2k-1} + \left(1 + \frac{2}{k}\right) a_{2k} + \cdots \\
&\quad + \left(1 + \frac{k-3}{k}\right) a_{3k-5} + \left(1 + \frac{k-2}{k}\right) a_{3k-4} + \cdots \\
&\quad + \sum_{n=2}^{2k-2} a_n \\
&\geq \frac{1}{k} (a_{2k-1} + a_{k+1} - a_{k-1}) + \frac{2}{k} (a_{2k} + a_{k+2} - a_{k-2}) \\
&\quad + \cdots + \frac{k-2}{k} (a_{3k-4} + a_{2k-2} - a_2) + \sum_{n=2}^{\infty} a_n \\
&= \sum_{j=0}^{k-3} \frac{j+1}{k} (a_{2k+j-1} + a_{k+j+1} - a_{k-j-1}) + \sum_{n=2}^{\infty} a_n \\
&\geq \sum_{n=2}^{\infty} a_n
\end{aligned}$$

with the following condition

$$\sum_{j=0}^{k-3} \frac{j+1}{k} (a_{2k+j-1} + a_{k+j+1} - a_{k-j-1}) \geq 0.$$

Thus, we observe that the function $\omega(z)$ defined by (2.3) satisfies $\omega(z)$ is analytic in \mathbb{U} with $\omega(0) = 0$, $|\omega(z)| < 1$ ($z \in \mathbb{U}$). This completes the proof of the theorem. \square

Remark 2.1. Taking $k = 2$ in Theorem 2.1, we have Theorem B by Silverman [4].

Example 2.1. Let us define

$$(2.5) \quad f(z) = z - \frac{37}{1200}z^2 - \frac{1}{18}z^3 - \frac{1}{48}z^4 - \frac{1}{100}z^5$$

and

$$(2.6) \quad f_3(z) = z - \frac{1}{3}z^3$$

with $k = 3$ in Theorem 2.1. Since $f(z)$ satisfies

$$\sum_{n=2}^{\infty} na_n = \frac{217}{600} < 1,$$

we have $f(z) \in \mathcal{T}^*$. Furthermore, $f(z)$ satisfies,

$$\frac{1}{3}(a_5 + a_4 - a_2) = \frac{1}{3} \left(\frac{1}{100} + \frac{1}{48} - \frac{37}{1200} \right) = 0.$$

Thus, $f(z)$ satisfies the conditions in Theorem 2.1 with $k = 3$.
If we take $\lambda = 2$, then we have

$$\int_0^{2\pi} |f(z)|^2 d\theta \leq 2\pi r^2 \left(1 + \frac{1}{9}r^4 \right) < \frac{20}{9}\pi = 6.9813 \dots$$

Corollary 2.1. Let $f(z) \in \mathcal{T}^*$, $0 < \lambda \leq 2$, and $f_k(z) = z - \frac{z^k}{k}$ ($k \geq 2$). If $f(z)$ satisfies (2.1) for $k \geq 3$, then, for $z = re^{i\theta}$ ($0 < r < 1$),

$$(2.6) \quad \int_0^{2\pi} |f(z)|^\lambda d\theta \leq 2\pi r^\lambda \left(1 + \frac{1}{k^2}r^{2(k-1)} \right)^{\frac{\lambda}{2}} < 2\pi \left(1 + \frac{1}{k^2} \right)^{\frac{\lambda}{2}}.$$

Proof. It follows that

$$\int_0^{2\pi} |f_k(z)|^\lambda d\theta = \int_0^{2\pi} |z|^\lambda \left| 1 - \frac{z^{k-1}}{k} \right|^\lambda d\theta.$$

Applying Hölder inequality for $0 < \lambda < 2$, we obtain that

$$\begin{aligned}
 \int_0^{2\pi} |z|^\lambda \left| 1 - \frac{z^{k-1}}{k} \right|^\lambda d\theta &\leq \left(\int_0^{2\pi} (|z|^\lambda)^{\frac{2}{2-\lambda}} d\theta \right)^{\frac{2-\lambda}{2}} \left(\int_0^{2\pi} \left(\left| 1 - \frac{z^{k-1}}{k} \right|^\lambda \right)^{\frac{2}{\lambda}} d\theta \right)^{\frac{\lambda}{2}} \\
 &= \left(\int_0^{2\pi} |z|^{\frac{2\lambda}{2-\lambda}} d\theta \right)^{\frac{2-\lambda}{2}} \left(\int_0^{2\pi} \left| 1 - \frac{z^{k-1}}{k} \right|^2 d\theta \right)^{\frac{\lambda}{2}} \\
 &= \left(2\pi r^{\frac{2\lambda}{2-\lambda}} \right)^{\frac{2-\lambda}{2}} \left(2\pi \left(1 + \frac{1}{k^2} r^{2(k-1)} \right) \right)^{\frac{\lambda}{2}} \\
 &= 2\pi r^\lambda \left(1 + \frac{1}{k^2} r^{2(k-1)} \right)^{\frac{\lambda}{2}} \\
 &< 2\pi \left(1 + \frac{1}{k^2} \right)^{\frac{\lambda}{2}}.
 \end{aligned}$$

Further, it is clear for $\lambda = 2$. □

For the generalization of Theorem C by Silverman [4], we have

Theorem 2.2. Let $f(z) \in \mathcal{T}^*$, $\lambda > 0$, and $f_k(z) = z - \frac{z^k}{k}$ ($k \geq 2$). Then, for $z = re^{i\theta}$ ($0 < r < 1$),

$$(2.7) \quad \int_0^{2\pi} |f'(z)|^\lambda d\theta \leq \int_0^{2\pi} |f'_k(z)|^\lambda d\theta.$$

Proof. For $f(z) \in \mathcal{T}^*$, it is sufficient to show that

$$(2.8) \quad 1 - \sum_{n=2}^{\infty} n a_n z^{n-1} < 1 - z^{k-1}.$$

Let us define the function $\omega(z)$ by

$$(2.9) \quad 1 - \sum_{n=2}^{\infty} n a_n z^{n-1} = 1 - \omega(z)^{k-1},$$

or, by

$$\omega(z)^{k-1} = \sum_{n=2}^{\infty} n a_n z^{n-1}.$$

Since $f(z)$ satisfies

$$\sum_{n=2}^{\infty} n a_n \leq 1,$$

the function $\omega(z)$ is analytic in \mathbb{U} , $\omega(0) = 0$, and $|\omega(z)| < 1$ ($z \in \mathbb{U}$). □

Remark 2.2. If we take $k = 2$ in Theorem 2.2, then we have Theorem C by Silverman [4].

Using Hölder inequality for Theorem 2.2, we have

Corollary 2.2. Let $f(z) \in \mathcal{T}^*$, $0 < \lambda \leq 2$, and $f_k(z) = z - \frac{z^k}{k}$ ($k \geq 2$). Then, for $z = re^{i\theta}$ ($0 < r < 1$),

$$\int_0^{2\pi} |f'(z)|^\lambda d\theta \leq 2\pi(1 + r^{2(k-1)})^{\frac{\lambda}{2}} < 2^{\frac{2+\lambda}{2}} \pi.$$

3 Integral means for functions in the class \mathcal{C}

In this section, we discuss the integral means for functions in the class \mathcal{C} .

Theorem 3.1. Let $f(z) \in \mathcal{C}$, $\lambda > 0$, and $f_k(z) = z - \frac{z^k}{k^2}$ ($k \geq 2$). If $f(z)$ satisfies

$$(3.1) \quad \sum_{j=2}^{k-1} \frac{(k+j)(k-j)}{k^2} (a_{2k-j} - a_j) \geq 0$$

for $k \geq 3$, then, for $z = re^{i\theta}$ ($0 < r < 1$),

$$(3.2) \quad \int_0^{2\pi} |f(z)|^\lambda d\theta \leq \int_0^{2\pi} |f_k(z)|^\lambda d\theta.$$

Proof. For the proof, we need to show that

$$(3.3) \quad 1 - \sum_{n=2}^{\infty} a_n z^{n-1} < 1 - \frac{z^{k-1}}{k^2}$$

by Theorem A. Define the function $\omega(z)$ by

$$(3.4) \quad 1 - \sum_{n=2}^{\infty} a_n z^{n-1} = 1 - \frac{1}{k^2} \omega(z)^{k-1},$$

or by

$$(3.5) \quad \omega(z)^{k-1} = k^2 \left(\sum_{n=2}^{\infty} a_n z^{n-1} \right).$$

Therefore, we have to show that

$$\sum_{n=2}^{\infty} a_n \leq \frac{1}{k^2} \left(\sum_{n=2}^{\infty} n^2 a_n \right).$$

Using the same technique as in the proof of Theorem 2.1, we see that

$$\begin{aligned} \frac{1}{k^2} \left(\sum_{n=2}^{\infty} n^2 a_n \right) &\geq \sum_{j=2}^{k-1} \frac{(k+j)(k-j)}{k^2} (a_{2k-j} - a_j) + \sum_{n=2}^{\infty} a_n \\ &\geq \sum_{n=2}^{\infty} a_n. \end{aligned}$$

□

Example 3.1. Consider the functions

$$(3.6) \quad f(z) = z - \frac{1}{40}z^2 - \frac{1}{18}z^3 - \frac{1}{40}z^4$$

and

$$(3.7) \quad f_3(z) = z - \frac{1}{9}z^3$$

with $k = 3$ in Theorem 3.1. Then we have that

$$\sum_{n=2}^{\infty} n^2 a_n = \frac{4}{40} + \frac{9}{18} + \frac{16}{40} = 1$$

which implies $f(z) \in \mathcal{C}$, and that

$$\frac{5}{9}(a_4 - a_2) = 0.$$

Thus $f(z)$ satisfies the conditions of Theorem 3.1. If we make $\lambda = 2$, then we see that

$$\int_0^{2\pi} |f(z)|^2 d\theta \leq 2\pi r^2 \left(1 + \frac{1}{81}r^4 \right) < \frac{164}{81}\pi = 6.3607 \dots$$

Corollary 3.1. Let $f(z) \in \mathcal{C}$, $0 < \lambda \leq 2$, and $f_k(z) = z - \frac{z^k}{k^2}$ ($k \geq 2$). If $f(z)$ satisfies the condition (3.1) for $k \geq 3$, then, for $z = re^{i\theta}$ ($0 < r < 1$),

$$\begin{aligned} (3.8) \quad \int_0^{2\pi} |f(z)|^\lambda d\theta &\leq 2\pi r^\lambda \left(1 + \frac{1}{k^4} r^{2(k-1)} \right)^{\frac{\lambda}{2}} \\ &< 2\pi \left(1 + \frac{1}{k^4} \right)^{\frac{\lambda}{2}}. \end{aligned}$$

Further, we may have

Theorem 3.2. Let $f(z) \in \mathcal{C}$, $\lambda > 0$, and $f_k(z) = z - \frac{z^k}{k^2}$ ($k \geq 2$). If $f(z)$ satisfies

$$(3.9) \quad \sum_{j=2}^{2k-2} j(k-j)a_j \leq 0,$$

then, for $z = re^{i\theta}$ ($0 < r < 1$),

$$(3.10) \quad \int_0^{2\pi} |f'(z)|^\lambda d\theta \leq \int_0^{2\pi} |f'_k(z)|^\lambda d\theta.$$

Example 3.2. Take the functions

$$(3.11) \quad f(z) = z - \frac{1}{24}z^2 - \frac{1}{18}z^3 - \frac{1}{48}z^4$$

and

$$(3.12) \quad f_3(z) = z - \frac{1}{9}z^3$$

with $k = 3$ in Theorem 3.2. Since

$$\sum_{n=2}^{\infty} n^2 a_n = \frac{4}{24} + \frac{9}{18} + \frac{16}{48} = \frac{5}{6} < 1$$

and

$$2(3-2)a_2 + 3(3-3)a_3 + 4(3-4)a_4 = \frac{1}{12} - \frac{1}{12} = 0,$$

$f(z)$ satisfies the conditions in Theorem 3.2. If we take $\lambda = 2$, then we have

$$\int_0^{2\pi} |f'(z)|^2 d\theta \leq 2\pi \left(1 + \frac{1}{9}r^4\right) < \frac{20}{9}\pi.$$

Corollary 3.2. Let $f(z) \in \mathcal{C}$, $0 < \lambda \leq 2$, and $f_k(z) = z - \frac{z^k}{k^2}$ ($k \geq 2$). If $f(z)$ satisfies the condition (3.9) for $k \geq 2$, then, for $z = re^{i\theta}$ ($0 < r < 1$),

$$\int_0^{2\pi} |f'(z)|^\lambda d\theta \leq 2\pi \left(1 + \frac{1}{k}r^{2(k-1)}\right)^{\frac{\lambda}{2}} < 2\pi \left(1 + \frac{1}{k}\right)^{\frac{\lambda}{2}}.$$

4 Appendix

For analytic functions $h(z)$ and $g(z)$, Hölder inequality gives that, for $z = re^{i\theta}$ ($0 < r < 1$),

$$(4.1) \quad \int_0^{2\pi} |h(z)g(z)| d\theta \leq \left(\int_0^{2\pi} |h(z)|^p d\theta\right)^{\frac{1}{p}} \left(\int_0^{2\pi} |g(z)|^q d\theta\right)^{\frac{1}{q}}$$

with $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. It follows from (4.1) that

$$(4.2) \quad \int_0^{2\pi} |h(z)|^p d\theta \geq \frac{\left(\int_0^{2\pi} |h(z)g(z)| d\theta \right)^p}{\left(\int_0^{2\pi} |g(z)|^q d\theta \right)^{\frac{p}{q}}}.$$

Letting $p = \frac{\lambda}{2}$, $q = \frac{\lambda}{\lambda-2}$, $\lambda > 2$, we observe that the function $f(z)$ in the class \mathcal{T} satisfies

$$(4.3) \quad \begin{aligned} \int_0^{2\pi} |f(z)|^\lambda d\theta &= \int_0^{2\pi} (|f(z)|^2)^{\frac{\lambda}{2}} d\theta \\ &\geq \frac{\left(\int_0^{2\pi} |f(z)|^2 d\theta \right)^{\frac{\lambda}{2}}}{\left(\int_0^{2\pi} d\theta \right)^{\frac{\lambda-2}{2}}} \\ &= (2\pi)^{\frac{2-\lambda}{2}} \left\{ 2\pi \left(r^2 + \sum_{n=2}^{\infty} a_n^2 r^{2n} \right) \right\}^{\frac{\lambda}{2}} \\ &= 2\pi r^\lambda \left(1 + \sum_{n=2}^{\infty} a_n^2 r^{2(n-1)} \right)^{\frac{\lambda}{2}}. \end{aligned}$$

Further, when $\lambda = 2$, we see that, for $z = re^{i\theta}$ ($0 < r < 1$),

$$\begin{aligned} \int_0^{2\pi} |f(z)|^2 d\theta &= 2\pi r^2 \left(1 + \sum_{n=2}^{\infty} a_n^2 r^{2(n-1)} \right) \\ &< 2\pi \left(1 + \sum_{n=2}^{\infty} a_n^2 \right). \end{aligned}$$

Therefore, we conclude that

Theorem 4.1. *Let $f(z) \in \mathcal{T}$ and $\lambda \geq 2$. Then, for $z = re^{i\theta}$ ($0 < r < 1$),*

$$(4.4) \quad \int_0^{2\pi} |f(z)|^\lambda d\theta \geq 2\pi r^\lambda \left(1 + \sum_{n=2}^{\infty} a_n^2 r^{2(n-1)} \right)^{\frac{\lambda}{2}}.$$

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Shigeyoshi Owa :
Department of mathematics
Kinki University
Higashi-Osaka, Osaka 577-8502
Japan
e-mail: owa@math.kindai.ac.jp

Daisuke Yagi :
Department of mathematics
Kinki University
Higashi-Osaka, Osaka 577-8502
Japan

Nicolae N. Pascu :
Department of Mathematics
Transilvania University of Brasov
R-2200 Brasov
Romania
e-mail: n.pascu@unitbv.ro